Lecture 16: October 28

Hodge classes. Today, I want to present a nice application of Schmid's results to the study of Hodge classes. Let me first recallt the definition.

Definition 16.1. Let $H_{\mathbb{Z}}$ be a $\mathbb{Z}$-Hodge structure of even weight $2k$; this means that $H_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module, and

$$H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=2k} H^{p,q}$$

is a Hodge structure of weight $2k$ with $H^{q,p} = \overline{H^{p,q}}$. A *Hodge class* is a class $v \in H_{\mathbb{Z}}$ whose image lies in $H^{k,k}$.

Note. Equivalently, a class $v \in H_{\mathbb{Z}}$ is a Hodge class iff its image in $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ lies in the subspace

$$F^k H = \bigoplus_{p \geq k} H^{p,2k-p}.$$

The reason is that v is real, and so its Hodge decomposition

$$v = \sum_{p+q=2k} v^{p,q}$$

has the property that $v^{q,p} = \overline{v^{p,q}}$. Now $v \in F^k H$ means that $v^{p,q} = 0$ for $p \leq k-1$, and therefore also for $p \geq k+1$. But then $v \in H^{k,k}$, and so v is a Hodge class.

Note. We can always reduce to the case $k = 0$ by considering the Tate twist $H_{\mathbb{Z}}(k)$. The underlying $\mathbb{Z}$-module is $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}(k)$, where $\mathbb{Z}(k) = (2\pi i)^k \mathbb{Z} \subseteq \mathbb{C}$. The Hodge structure $H(k)$ has weight $2k - 2k = 0$, and $H(k)^{0,0} = H^{k,k}$, so Hodge classes of type (k,k) in $H_{\mathbb{Z}}$ are the same thing as Hodge classes of type $(0,0)$ in $H_{\mathbb{Z}}(k)$.

Hodge classes are interesting because of their relation with algebraic cycles. If X is a smooth projective variety (or a compact Kähler manifold), and $Z \subseteq X$ an algebraic (or analytic) subvariety of codimension k, then the cycle class $[Z] \in H^{2k}(X, \mathbb{Z})$ is a Hodge class. The *Hodge conjecture* predicts that on any smooth projective variety, a nonzero multiple of every Hodge class is a linear combination of cycle classes. (This is known to be false for compact Kähler manifolds in general.)

The Hodge conjecture also makes some suprising predictions for families of smooth projective varieties. Let $f \colon X \to B$ be a projective morphism between smooth algebraic varieties, with B say quasiprojective. Suppose we have a Hodge class $h \in H^{2k}(X_{b_0}, \mathbb{Z})$ in the fiber over a point $b_0 \in B$. We can transport h to a class in $H^{2k}(X_b, \mathbb{Z})$ over nearby points $b \in B$, and ask for which $b \in B$ the resulting class is again a Hodge class. It is not hard to see that the set of nearby points with this property is (the germ of) an analytic subset of B. But if the Hodge conjecture is true, then this set should actually be (the germ of) an *algebraic* subset of B. Indeed, the condition for a given class to be a linear combination of cycle classes of algebraic subvarieties in an algebraic condition on b, because subvarieties of the fibers are parametrized by open subvarieties of certain Hilbert schemes, which are algebraic varieties.

More generally, we can look at the set of pairs (b, h), where $b \in B$ is a point, and $h \in H^{2k}(X_b, \mathbb{Z})$ is a Hodge class on the fiber. This set is called the *locus of Hodge classes*; it is again not hard to show that it has the structure of a complex space. If the Hodge conjecture is true, then the locus of Hodge classes should be a countable union of *algebraic* varieties. In a famous paper from the 1990's, Cattani, Deligne, and Kaplan proved that this is true, independently of the Hodge conjecture. Before stating their result more precisely, let us first note the following finiteness property of Hodge classes.

Example 16.2. Let $H_{\mathbb{Z}}$ be a $\mathbb{Z}$-Hodge structure of weight 0, and suppose that the Hodge structure on

$$H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=0} H^{p,q}$$

is polarized by a hermitian pairing $h \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$. The Hodge norm of a Hodge class $v \in H_{\mathbb{Z}}$ equals $h(v,v)$, and so for any $K \geq 0$, the set

$$\left\{ v \in H_{\mathbb{Z}} \;\middle|\; v \text{ is a Hodge class and } h(v,v) \leq K \right\}$$

is discrete and compact, hence finite. In other words, the set of Hodge classes of bounded self-intersection number is always finite.

The result of Cattani, Deligne, and Kaplan works not just for families of smooth projective varieties, but for any polarized variation of $\mathbb{Z}$-Hodge structure of weight 0 over a smooth algebraic variety X. This means that $\mathscr{V}$ is a polarized variation of Hodge structure of weight 0, with polarization $h \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^{\infty}_X$, and $\mathscr{V}_{\mathbb{Z}}$ is a local system of $\mathbb{Z}$-modules such that $\mathscr{V} = \mathscr{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathscr{O}_X$. Here is the statement of the Cattani-Deligne-Kaplan theorem.

Theorem 16.3. *Let X be a smooth quasi-projective algebraic variety, and $\mathscr{V}$ a polarized variation of $\mathbb{Z}$-Hodge structure of weight 0. For any $K \geq 0$, consider the set of pairs (x,v), where $x \in X$ and $v \in \mathscr{V}_{\mathbb{Z},x}$ is a Hodge class of type $(0,0)$ with $h_x(v,v) \leq K$. Then this set is the set of points of a quasi-projective algebraic variety, which is moreover finite and proper over X.*

Following Cattani, Deligne, and Kaplan, we will call the set

$$\operatorname{Hdg}(\mathscr{V}) = \left\{ (x,v) \;\middle|\; x \in X, \text{ and } v \in \mathscr{V}_{\mathbb{Z},x} \text{ is a Hodge class of type } (0,0) \right\}$$

the *locus of Hodge classes* of the given variation of Hodge structure; according to Theorem 16.3, it is a countable union of algebraic varieties. The subset

$$\operatorname{Hdg}_K(\mathscr{V}) = \left\{ (x,v) \in \operatorname{Hdg}(\mathscr{V}) \;\middle|\; h_x(v,v) \leq K \right\}$$

is called the *locus of Hodge classes of self-intersection number* $\leq K$; according to Theorem 16.3, it consists of finitely many connected components of $\operatorname{Hdg}(\mathscr{V})$, and the projection $\operatorname{Hdg}_K(\mathscr{V}) \to X$ is finite and proper.

Reduction to Schmid's results. My goal is to explain the proof of Theorem 16.3 when $\dim X = 1$, as an application of Schmid's results. Let us first check that $\operatorname{Hdg}(\mathscr{V})$ is the set of points of a closed analytic subvariety of a complex manifold. For simplicity, let me assume from now on that $\mathscr{V}_{\mathbb{Z}}$ is torsion-free. Consider the *étalé space* of the local system,

$$\text{Ét}(\mathscr{V}_{\mathbb{Z}}) = \left\{ (x,v) \;\middle|\; x \in X, \text{ and } v \in \mathscr{V}_{\mathbb{Z},x} \right\},$$

which is the disjoint union of all the stalks, with the finest topology that makes every section of $\mathscr{V}_{\mathbb{Z}}$ continuous. Since $\mathscr{V}_{\mathbb{Z}}$ is locally constant, $\text{Ét}(\mathscr{V}_{\mathbb{Z}})$ is locally isomorphic to a product, and therefore an infinite-sheeted covering space of X. In particular, $\text{Ét}(\mathscr{V}_{\mathbb{Z}})$ is again a complex manifold. Denote by $\mathbb{V}$ the holomorphic vector bundle whose sheaf of sections in $\mathscr{V}$, and by $F^p\mathbb{V}$ the subbundle corresponding to the subsheaf $F^p\mathscr{V}$. The morphism of sheaves $\mathscr{V}_{\mathbb{Z}} \hookrightarrow \mathscr{V}$ gives rise to an embedding

$$\text{Ét}(\mathscr{V}_{\mathbb{Z}}) \hookrightarrow \mathbb{V},$$

and the locus of Hodge classes is exactly the intersection $\text{Ét}(\mathscr{V}_{\mathbb{Z}}) \cap F^0\mathbb{V}$. This shows that $\operatorname{Hdg}(\mathscr{V})$ is the set of points of an analytic subvariety of the complex manifold $\text{Ét}(\mathscr{V}_{\mathbb{Z}})$. Since the pairing h is flat, it is constant on every connected component of $\text{Ét}(\mathscr{V}_{\mathbb{Z}})$; therefore $\operatorname{Hdg}_K(\mathscr{V})$ is a union of connected components of $\operatorname{Hdg}(\mathscr{V})$, and therefore again an analytic subvariety. Example 16.2 shows that the projection $\mathrm{Hdg}_K(\mathscr{V}) \to X$ has finite fibers.

Here is the general idea for proving that $\mathrm{Hdg}_K(\mathscr{V})$ is an algebraic variety. Choose a projective compactification $\bar{X} \supseteq X$, and show that $\mathrm{Hdg}_K(\mathscr{V})$ can be extended to a complex space that is proper and finite over $\bar{X}$. Since $\bar{X}$ is projective, it follows from Chow's theorem that the extension is a projective algebraic variety; but then $\mathrm{Hdg}_K(\mathscr{V})$, being the preimage of X, must be quasi-projective. The extension is constructed with the help of Schmid's results, but the job becomes more manageable after a few initial reductions. First, we know from Corollary 8.4 that the local monodromy of $\mathscr{V}_{\mathbb{Z}}$ around each point of $\bar{X} \setminus X$ is quasi-unipotent. One can find a finite covering space $\pi \colon Y \to X$ such that the pullback $\pi^* \mathscr{V}_{\mathbb{Z}}$ of the local system has *unipotent* local monodromy at each point of $\bar{Y} \setminus Y$.

Example 16.4. One way to construct $\pi \colon Y \to X$ is by using the following fact: If all eigenvalues of an integer matrix are roots of unity, and if the matrix is congruent to the identity modulo some prime number $p \geq 3$, then the matrix is actually unipotent. So it suffices to take a finite covering space with the property that the local system $\pi^* \mathscr{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is trivial. Because of Corollary 8.4, this implies that $\pi^* \mathscr{V}_{\mathbb{Z}}$ has unipotent local monodromy.

The finite covering space extends uniquely to a finite morphism $\pi \colon \bar{Y} \to \bar{X}$ between projective compactifications. If we can show that $\mathrm{Hdg}_K(\pi^* \mathscr{V})$ is an algebraic variety that is finite and proper over Y, then by composing with π, the same thing is true for $\mathrm{Hdg}_K(\mathscr{V})$ itself. After replacing $\mathscr{V}$ by $\pi^* \mathscr{V}$, we can therefore assume without loss of generality that the local monodromy of $\mathscr{V}_{\mathbb{Z}}$ around each point of $\bar{X} \setminus X$ is unipotent.

Next, the problem of constructing an extension of $\mathrm{Hdg}_K(\mathscr{V})$ is local near each point of $\bar{X} \setminus X$. We therefore need to consider a polarized variation of $\mathbb{Z}$-Hodge structure on the punctured disk Δ^*, with unipotent local monodromy. As usual, we denote by $\mathscr{V}$ the underlying vector bundle, by $F^p \mathscr{V}$ the Hodge bundles, and by h the polarization. Let V be the vector space of all flat sections of $\exp^* \mathscr{V}$ on the halfspace $\tilde{\mathbb{H}}$. We have $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, where $V_{\mathbb{Z}}$ is the (free) $\mathbb{Z}$-module of sections of the local system $\exp^* \mathscr{V}_{\mathbb{Z}}$. The monodromy transformation $T \in \mathrm{End}(V_{\mathbb{Z}})$ is defined over $\mathbb{Z}$ and unipotent. Therefore $T = e^{2\pi i R}$, with $R \in \mathrm{End}(V)$ nilpotent; in the notation of Lecture 9, this means that $R_N = R$ and $R_S = 0$. Let $\tilde{\mathscr{V}}$ be the canonical extension for the interval $[0,1)$, with the distinguished trivialization

$$\mathscr{O}_{\Delta} \otimes_{\mathbb{C}} V \cong \tilde{\mathscr{V}}$$

in which the connection is given by

$$\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv.$$

The vector bundle $\mathbb{V}$ is therefore trivial, with $\mathbb{V} \cong \Delta^* \times V$. In this setting, the étalé space $\acute{\mathrm{E}}\mathrm{t}(\mathscr{V}_{\mathbb{Z}})$ has the following concrete description: it is the image of the holomorphic mapping

$$\tilde{\mathbb{H}} \times V_{\mathbb{Z}} \to \Delta^* \times V, \quad (z, v) \mapsto \Big(e^z, e^{-zR} v \Big).$$

Let $\Phi \colon \tilde{\mathbb{H}} \to D$ be the period mapping of the variation of Hodge structure, and $\Psi \colon \Delta \to \check{D}$ be the holomorphic mapping from Theorem 9.1; here $\Psi(e^z) = e^{-zR} \Phi(z)$. At each point $t \in \Delta^*$, the fiber of the Hodge bundle $F^p \mathscr{V}$ is then $F^p_{\Psi(t)}$, in the above trivialization.

What do Hodge classes look like in this setting? By construction, a pair (t, w) belongs to $\acute{\mathrm{E}}\mathrm{t}(\mathscr{V}_{\mathbb{Z}})$ if $t = e^z$ and $w = e^{-zR} v$ for some $z \in \tilde{\mathbb{H}}$ and some $v \in V_{\mathbb{Z}}$; the pair is a Hodge class if $w \in F^0_{\Psi(t)}$. The latter is of course equivalent to $v \in F^0_{\Phi(z)}$. The

Hodge norm of such a class is $h(v, v)$, and so the condition for $(t, w) \in \mathrm{Hdg}_K(\mathscr{V})$ is that $h(v, v) \leq K$. Our goal is to construct an extension of $\mathrm{Hdg}_K(\mathscr{V})$ that is finite and proper over Δ. We therefore need to understand the possible limits of sequences of Hodge classes. The precise technical result that we are going to prove is the following.

Proposition 16.5. *With notation as above, suppose that $z_n \in \tilde{\mathbb{H}}$ is a sequence of points with bounded imaginary parts, such that $t_n = e^{z_n} \to 0$. Also suppose that $v_n \in V_{\mathbb{Z}}$ is a sequence of integral classes with $h(v_n, v_n) \leq K$, such that $v_n \in F^0_{\Phi(z_n)}$ for every $n \in \mathbb{N}$. Then after passing to a subsequence, v_n is constant, and the constant value belongs to $F^0_{\Psi(0)} \cap \ker R$.*

We are going to construct the desired extension of $\mathrm{Hdg}_K(\mathscr{V})$ after proving this technical result. The main ingredient is Theorem 10.2 and Theorem 10.3.

Proof of the technical result. I am going to divide the proof into four steps.

Step 1. We use the bound on the self-intersection to deduce that $v_n \in W_0$. Here $W_\bullet$ is the monodromy weight filtration of the nilpotent operator $R \in \mathrm{End}(V)$. In our setting, the monodromy weight filtration is actually defined over $\mathbb{Q}$, because

$$2\pi i R = \log T = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell!} (T - \mathrm{id})^\ell$$

is an endomorphism of $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. As in Lecture 9, we choose a semisimple endomorphism $H \in \mathrm{End}(V)$ with $[H, R] = -2R$, such that $W_j = E_j(H) \oplus W_{j-1}$ for all $j \in \mathbb{Z}$. Recall from Theorem 10.2 that

$$\hat{F} = \lim_{t \to 0} e^{\frac{1}{2} \log L(t) H} F_{\Psi(t)}$$

exists, and that $e^{-\frac{1}{2}R}\hat{F}$ is the Hodge filtration of a polarized Hodge structure of weight 0. Define the sequence of operators

$$g_n = e^{\frac{1}{2} \log L(t_n) H} \in G_{\mathbb{R}}.$$

Note that each g_n is an endomorphism of $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. The operator g_n acts on the eigenspace $E_\ell(H)$ as multiplication by $L(t_n)^{\ell/2}$. Since the imaginary parts of z_n are bounded, we have (by Exercise 10.1)

$$e^{-\frac{1}{2}R}\hat{F} = \lim_{n \to \infty} e^{\frac{1}{2} \log L(t_n) H} e^{\frac{1}{2}(\bar{z}_n - z_n) R} F_{\Phi(z_n)} = \lim_{n \to \infty} g_n F_{\Phi(z_n)}.$$

In terms of Hodge norms, the bound $h(v_n, v_n) \leq K$ means that

$$\|g_n v_n\|^2_{g_n \Phi(z_n)} = \|v_n\|^2_{\Phi(z_n)} = h(v_n, v_n) \leq K.$$

Because the Hodge norm at the point $g_n \Phi(z_n)$ converges to the Hodge norm for the Hodge structure $e^{\frac{1}{2}R}\hat{F}$, we conclude (as in Lecture 11) that the sequence $g_n v_n \in V_{\mathbb{R}}$ is bounded. This implies that $v_n \in W_0$ for all but finitely many values of n. Indeed, suppose that, for some $\ell \geq 1$, we had $v_n \in W_\ell$ for infinitely many values of n. Since $g_n v_n$ is bounded, and since $L(t_n)^\ell$ is going to infinity, it follows that the component of v_n in the eigenspace $E_\ell(H)$ must be going to zero. But

$$E_\ell(H) \cong W_\ell / W_{\ell-1},$$

and since $v_n \in V_{\mathbb{Z}}$, the projection of v_n into $W_\ell / W_{\ell-1}$ takes values in a discrete set, hence must be equal to zero after all. So after omitting finitely many terms from the beginning of the sequence, we can assume that $v_n \in W_0$. For the same reason, the component of v_n in $E_0(H)$ must then take values in a finite set, and so after replacing v_n by a subsequence, we can assume that $(v_n)_0 \in E_0(H)$ is constant.

Step 2. Since $v_n \in W_0$, we have $Rv_n \in W_{-2}$. We now prove that actually $Rv_n \in W_{-3}$. The boundedness of the sequence $g_n v_n$ means that, after passing to a subsequence, the limit

$$v = \lim_{n\to\infty} g_n v_n \in V_{\mathbb{R}}$$

exists. Of course, $v \in W_0$, and since g_n acts on $E_0(H)$ as the identity, the $E_0(H)$-component of v is equal to $(v_n)_0$ (which we already know to be constant). Since $v_n \in F^0_{\Phi(z_n)}$, it is also clear that

$$v \in \lim_{n\to\infty} g_n F^0_{\Phi(z_n)} = e^{-\frac{1}{2}R} F^0.$$

The following lemma, which you can think of as being a toy case of the general result we are trying to prove, now gives us $v \in E_0(H) \cap \ker R$.

Lemma 16.6. *Suppose that $V_{\mathbb{R}}$ has a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight 0. If a vector $v \in V_{\mathbb{R}}$ belongs to $W_0(Y) \cap e^{-\frac{1}{2}Y} F^0$, then $Yv = Hv = 0$ and $v \in F^0$.*

Proof. Let me spell out the assumptions in detail, before giving the proof. First, the complexification $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ should have an action by $\mathfrak{sl}_2(\mathbb{C})$, such that X and Y are purely imaginary, and H is real. Moreover, the hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ should be compatible with the $\mathfrak{sl}_2(\mathbb{C})$-action. Second, there should be a decreasing filtration $F^\bullet$ on V, such that $YF^p \subseteq F^{p-1}$ and $HF^p \subseteq F^{p-1}$ for all $p \in \mathbb{Z}$. Moreover, the filtration $e^{-\frac{1}{2}Y} F$ should define an $\mathbb{R}$-Hodge structure of weight 0 on V, which is polarized by the pairing h. We know from Theorem 14.1 that V is actually a polarized Hodge-Lefschetz structure of central weight 0; in particular, each eigenspace $E_j(H)$ has a real Hodge structure of weight j, with Hodge filtration $F^\bullet \cap E_j(H)$.

Now consider a vector $v \in V_{\mathbb{R}} \cap W_0(Y) \cap e^{-\frac{1}{2}Y} F^0$. Write $v = v_0 + v_{-1} + v_{-2} + \cdots$ for the decomposition into H-eigenspaces, with $v_j \in E_j(H)$; note that $v_j \in V_{\mathbb{R}}$, due to the fact that H is a real operator. Then

$$e^{\frac{1}{2}Y} v = v_0 + v_{-1} + \left(v_{-2} + \frac{1}{2} Y v_0\right) + \cdots \in F^0,$$

and since the filtration F is compatible with H, the individual summands belong to F^0. Therefore $v_0 \in F^0$, $v_{-1} \in F^0$, $v_{-2} + \frac{1}{2} Y v_0 \in F^0$, and so on. We can now proceed step by step to show that $Rv_0 = 0$ and $v_j = 0$ for $j \leq -1$.

(1) We know that $v_0 \in V_{\mathbb{R}} \cap E_0(H) \cap F^0$. As $E_0(H)$ has a real Hodge structure of weight 0, it follows that $v_0 \in E_0(H)^{0,0}$ is of type $(0,0)$.
(2) Similarly, $v_{-1} \in V_{\mathbb{R}} \cap E_{-1}(H) \cap F^0$, and because $E_{-1}(H)$ has a real Hodge structure of weight -1, it follows that $v_{-1} = 0$.
(3) Since $v_0 \in E_0(H)^{0,0}$, and since $Y\colon E_0(H) \to E_{-2}(H)(-1)$ is a morphism of Hodge structures, we get $Yv_0 \in E_{-2}(H)^{-1,-1}$. Therefore $v_{-2} \in V_{\mathbb{R}} \cap E_{-2}(H) \cap F^{-1}$, and because $E_{-2}(H)$ has a real Hodge structure of weight -2, we get $v_{-2} \in E_{-2}(H)^{-1,-1}$. Now $v_{-2} + \frac{1}{2} Y_0 v_0$ has type $(-1,-1)$ and also lies in F^0, and so it must be zero. Because v_{-2} is real and Yv_0 is purely imaginary, it follows that $v_{-2} = 0$ and $Yv_0 = 0$.
(4) Continuing in this way, one proves that $v_j = 0$ for every $j \leq -3$.

The conclusion is that $v = v_0$ belongs to $E_0(H) \cap \ker R \cap F^0$. ∎

In our setting, the lemma tells us that $v \in E_0(H)$ and $Rv = 0$. Recall that $v_n \in W_0$, and that the $E_0(H)$-component is constant and satisfies $(v_n)_0 = v$. Therefore $R(v_n)_0 = Rv = 0$, and therefore $Rv_n \in W_{-3}$.

Step 3. We can now prove that $Rv_n = 0$ for all but finitely many $n \in \mathbb{N}$. The proof is by contradiction. Suppose that $Rv_n \neq 0$ for infinitely many $n \in \mathbb{N}$. Fix a norm on the vector space V, and denote by u_n the unit vector in the direction of $g_n R v_n$ (which exists whenever $Rv_n \neq 0$). After passing to a subsequence, we can assume that $u = \lim_{n\to\infty} u_n \in V_{\mathbb{R}}$ exists. Of course, u is again a unit vector; since $Rv_n \in W_{-3}$, we also have $u \in W_{-3}$. I claim that moreover $u \in e^{-\frac{1}{2}R}\hat{F}^{-1}$. Here is the proof. From the fact that $v_n \in F^0_{\Phi(z_n)}$, we get

$$g_n R v_n \in g_n R F^0_{\Phi(z_n)} \subseteq g_n F^{-1}_{\Phi(z_n)} = F^{-1}_{g_n \Phi(z_n)}.$$

The subspaces on the right converge to $e^{-\frac{1}{2}R}\hat{F}^{-1}$, and so $u \in e^{-\frac{1}{2}R}\hat{F}^{-1}$ as desired. But now $u \in V_{\mathbb{R}} \cap W_{-3} \cap e^{-\frac{1}{2}R}\hat{F}^{-1}$ implies, as in the proof of the lemma, that $u = 0$, due to the fact that each eigenspace $E_j(H)$ has a real Hodge structure of weight j. This is a contradiction, and so we must have $Rv_n = 0$ for all but finitely many $n \in \mathbb{N}$ after all.

Exercise 16.1. Show that $u \in V_{\mathbb{R}} \cap W_{-3} \cap e^{-\frac{1}{2}R}\hat{F}^{-1}$ implies $u = 0$. (Hint: Start by proving that $V_{\mathbb{R}} \cap E_j(H) \cap \hat{F}^{-1} = 0$ for every $j \leq -3$.)